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The Elliptic Inequalities in the Lunar Theory.

By Ernest W. Brown, *Fellow of Christ's College, Cambridge.*

[*Continued from page* 263.]

The coefficients depending on the first power of the eccentricity having been obtained with sufficient accuracy, it is possible to proceed and obtain those depending on the square and higher powers of the same quantity. It is to be remembered that the value of c, the ratio of the two periods, was only the first term of an infinite series arranged in powers of the square of the eccentricity. Hence in obtaining the coefficients depending on higher powers of the eccentricity, it may be found necessary to have more terms of this series calculated.

V.

In the system of equations (4) put $p = 0$, there results

$$\Sigma_j \Sigma_q \{ (j, i, 0, q)\, A_{j,\,q} A_{j-i,\,q} + (i, 0)\, A_{j,\,q} A_{i-j-1,\,-q} + [i, 0]\, A_{j,\,q} A_{-i-j-1,\,-q} \} = 0$$

for all integral values of i, except $i = 0$. Now the suffix $\pm q$ denotes that the term to which it is attached involves the eccentricity raised to the positive power q, hence in this equation only even powers of the eccentricity will occur. Omitting terms which are of an order higher than Y_0^2, we have, in accordance with the notation previously used,

$$A_{j,\,1} = \varepsilon_j, \quad A_{j,\,-1} = \varepsilon_j'.$$

We now put

$$A_{j,\,0} = a_j + \delta a_j$$

instead of a_j, where δa_j denotes the new part depending on the eccentricity; a_j has the numerical value before assigned and δa_j is evidently of the order Y_0^2.

Hence q will only take the values $0, \pm 1$. Performing these substitutions we obtain

$$\begin{aligned}\Sigma_j\{(j,\ i,\ 0,\ 0)(a_j + \delta a_j)(a_{j-i} + \delta a_{j-i}) + (j,\ i,\ 0,\ 1)\,\varepsilon_j\varepsilon_{j-i} + (j,\ i,\ 0,\ -1)\,\varepsilon'_j\varepsilon'_{j-i}\}\\ + (i,\ 0)\,\Sigma_j\{(a_j + \delta a_j)(a_{i-j-1} + \delta a_{i-j-1}) + 2\varepsilon_j\varepsilon'_{i-j-1}\}\\ + [i,\ 0]\,\Sigma_j\{(a_j + \delta a_j)(a_{-i-j-1} + \delta a_{-i-j-1}) + 2\varepsilon_j\varepsilon'_{-i-j-1}\} = 0.\end{aligned}$$

If we omit the terms depending on Y_0^2 the equation becomes the same as that used by Dr. Hill to obtain a_i.

It is necessary to see in what way c is involved in the coefficients $(j,\ i,\ 0,\ 0)$, etc. Referring back to the general equations in Section I it was seen that c always occurred in one of the combinations $2i + cp$, $2j + cq$. The coefficients $(j,\ i,\ 0,\ 0)$, $(i,\ 0)$ and $[i,\ 0]$ will therefore be independent of c, while the two coefficients $(j,\ i,\ 0,\ \pm 1)$ will only involve this letter in their numerators. As the last series of coefficients only multiply quantities which are themselves of the order Y_0^2, it is sufficient for the purpose of obtaining $a_i + \delta a_i$ to this order to use the known part of the value of c which is independent of the eccentricity.

As the quantities a_i retain their previous signification, we obtain for the determination of δa_i the set of equations

$$\begin{aligned}\Sigma_j\{(j,\ i,\ 0,\ 0)(a_j\delta a_{j-i} + a_{j-i}\delta a_j) + (j,\ i,\ 0,\ 1)\,\varepsilon_j\varepsilon_{j-i} + (j,\ i,\ 0,\ -1)\,\varepsilon'_j\varepsilon'_{j-i}\\ + 2\,(i,\ 0)(a_{i-j-1}\delta a_j + \varepsilon_j\varepsilon'_{i-j-1}) + 2\,[i,\ 0]\,(a_{-i-j-1}\delta a_j + \varepsilon_j\varepsilon'_{-i-j-1})\} = 0\end{aligned}$$

omitting as before that for $i = 0$.

The labour of calculation is conducted in a way similar to that which was followed in the case of ε_i, ε'_i. The amount seems greater on account of the larger number of terms to be calculated; it is, however, not necessary to push the results to so many places of decimals. In addition, the coefficients $(j,\ i,\ 0,\ 0)$, $(i,\ 0)$ and $[i,\ 0]$ are given by Dr. Hill in Vol. I, pp. 245–6; his notations for them are respectively $(j,\ i)$, (i) and $[i]$. The terms in ε_i, ε'_i can be calculated exactly at the first approximation by using the values previously given for these quantities, and in obtaining the second approximation it is necessary only to find the changes in δa_i. I recall that $a_i = 1$ when $i = 0$, and therefore for that value of i, $\delta a_i = 0$.

The values of δa_i are given below to eight places of decimals. The calculations were made to nine places, and to that order a third approximation was found to be unnecessary.

$$\delta a_1 = +.03938\;170\;\;Y_0^2, \qquad \delta a_{-1} = +.01376\;519\;\;Y_0^2,$$
$$\delta a_2 = +.00046\;113\;\;Y_0^2, \qquad \delta a_{-2} = +.00002\;216\;\;Y_0^2,$$
$$\delta a_3 = +.00000\;473\;\;Y_0^2, \qquad \delta a_{-3} = +.00000\;026\;\;Y_0^2,$$
$$\delta a_4 = +.00000\;005\;\;Y_0^2, \qquad \delta a_{-4} = +.00000\;000\;\;Y_0^2.$$

For the sake of completeness the value of δa_0, the part of a_0 which depends on the square of the eccentricity is obtained. We have (Vol. I, p. 131)

$$\frac{\varkappa u}{(us)^{\frac{3}{2}}} = (D^2 + 2mD)\,u + \tfrac{3}{2}m^2\,(u + s).$$

Substitute in this equation

$$u = (a_0 + \delta a_0)\,\Sigma_i\{a_i + \delta a_i + \varepsilon_i\zeta^c + \varepsilon_i'\zeta^{-c}\}\zeta^{2i+1},$$
$$s = (a_0 + \delta a_0)\,\Sigma_i\{a_{-i-1} + \delta a_{-i-1} + \varepsilon'_{-i-1}\zeta^c + \varepsilon_{-i-1}\zeta^{-c}\}\zeta^{2i+1},$$

where a_0 has the numerical relation to $(\mu/n^2)^{\frac{1}{3}}$ given by Dr. Hill. Since the result of such a substitution ought to be an identity for all powers of ζ, consider the set of terms independent of $\zeta^{\pm c}$. Those on the right-hand side to the order Y_0^2 are, after division by $a_0 + \delta a_0$,

$$\Sigma_i[(a_i + \delta a_i)\{(2i + 1 + m)^2 + \tfrac{1}{2}m^2\} + \tfrac{3}{2}m^2\,(a_{-i-1} + \delta a_{-i-1})\,\zeta^{2i+1}].$$

Put $\zeta = 1$ and let

$$H = \Sigma_i a_i\{(2i + 1 + m)^2 + 2m^2\},$$
$$\delta H = \Sigma_i \delta a_i\{(2i + 1 + m)^2 + 2m^2\}.$$

The right-hand side then becomes $H + \delta H$ when $\zeta = 1$.

In the expressions for u, s just given, put $t = t_0$, i. e. $\zeta = 1$ except where it occurs to the index $\pm c$:

$$u = (a_0 + \delta a_0)\,\Sigma_i\{a_i + \delta a_i + \varepsilon_i\zeta^c + \varepsilon_i'\zeta^{-c}\},$$
$$s = (a_0 + \delta a_0)\,\Sigma_i\{a_i + \delta a_i + \varepsilon_i'\zeta^c + \varepsilon_i\zeta^{-c}\},$$

and substitute in $\varkappa u/(a_0 + \delta a_0)(us)^{\frac{3}{2}}$. Taking out those terms as before independent of $\zeta^{\pm c}$, we obtain as far as the order Y_0^2,

$$\begin{aligned}(a_0 + \delta a_0)^2\,\frac{u}{(us)^{\frac{3}{2}}} &= (\Sigma a_i + \Sigma\delta a_i + \Sigma\varepsilon_i\zeta^c + \Sigma\varepsilon_i'\zeta^{-c})^{-\frac{1}{2}}\,(\Sigma a_i + \Sigma\delta a_i + \Sigma\varepsilon_i'\zeta^c + \Sigma\varepsilon_i\zeta^{-c})^{-\frac{3}{2}}\\ &= \frac{1}{(\Sigma a_i)^2}\left(1 - \frac{1}{2}\,\frac{\Sigma\delta a_i}{\Sigma a_i} - \frac{1}{2}\,\frac{\Sigma\varepsilon_i}{\Sigma a_i}\,\zeta^c - \frac{1}{2}\,\frac{\Sigma\varepsilon_i'}{\Sigma a_i}\,\zeta^{-c} + \frac{3}{4}\,\frac{\Sigma\varepsilon_i.\Sigma\varepsilon_i'}{(\Sigma a_i)^2}\right)\\ &\quad\times\left(1 - \frac{3}{2}\,\frac{\Sigma\delta a_i}{\Sigma a_i} - \frac{3}{2}\,\frac{\Sigma\varepsilon_i'}{\Sigma a_i}\,\zeta^c - \frac{3}{2}\,\frac{\Sigma\varepsilon_i}{\Sigma a_i}\,\zeta^{-c} + \frac{15}{4}\,\frac{\Sigma\varepsilon_i.\Sigma\varepsilon_i'}{(\Sigma a_i)^2}\right),\end{aligned}$$

which gives

$$\frac{1}{(\Sigma a_i)^2}\left(1-\frac{2\Sigma\delta a_i}{\Sigma a_i}+\frac{3}{4}\,\frac{(\Sigma\varepsilon_i+\Sigma\varepsilon_i')^2+4\Sigma\varepsilon_i.\Sigma\varepsilon_i'}{(\Sigma a_i)^2}\right)$$

for the terms independent of $\zeta^{\pm c}$.

Let this expression be put equal to $J+\delta J$ where $J=1/(\Sigma a_i)^2$. Taking account of the value of $\varkappa$ which is $\mu(1+m)^2/n^2$, we obtain

$$a_0+\delta a_0=\left(\frac{\mu}{n^2}\right)^{\frac{1}{3}}\left[\frac{(J+\delta J)(1+m)^2}{H+\delta H}\right]^{\frac{1}{3}}.$$

Dr. Hill has obtained (Vol. I, p. 144)

$$a_0=\left(\frac{\mu}{n^2}\right)^{\frac{1}{3}}\left[\frac{J(1+m)^2}{H}\right]^{\frac{1}{3}}$$

$$=+.99909\ 31420\left(\frac{\mu}{n^2}\right)^{\frac{1}{3}}.$$

Hence

$$\frac{\delta a_0}{a_0}=\frac{1}{3}\cdot\frac{\delta J}{J}-\frac{1}{3}\cdot\frac{\delta H}{H}$$

$$=-\frac{2}{3}\,\frac{\Sigma\delta a_i}{\Sigma a_i}+\frac{(\Sigma\varepsilon_i+\Sigma\varepsilon_i')^2+4\Sigma\varepsilon_i.\Sigma\varepsilon_i'}{4(\Sigma a_i)^2}-\frac{\Sigma\delta a_i\{(2i+1+m)^2+2m^2\}}{3\Sigma a_i\{(2i+1+m)^2+2m^2\}}$$

Reducing to numbers by means of the known results,

$$\frac{\delta a_0}{a_0}=-.13311\ 28\ Y_0^2,$$

and therefore the common factor for the series in u, s to the order Y_0^2 is

$$a_0+\delta a_0=(+.99909\ 31420-.13299\ 21\ Y_0^2)\left(\frac{\mu}{n^2}\right)^{\frac{1}{3}}.$$

VI.

The next step forward requires us to obtain the series of coefficients $(i, \pm 2)$. In equation (4) put $p=\pm 2$ successively. It will be seen that the terms are all of the order Y_0^2 at least, and that to obtain them to this order we only need to know ε_i, ε_i' and the values of a_i; those of δa_i are not required. In this case q only takes the values 0, ± 1, ± 2, terms which involve a higher second suffix than ± 2 being omitted. It is convenient to denote $A_{i,\,2}$ and $A_{i,\,-2}$ by f_i and f_i' respectively on the same plan as before. The equations for the determination of f_i and f_{-i}' become

$$\begin{aligned}
&\Sigma_j \{ (j,\ i,\ 2,\ 2)\, a_{j-i} f_j + (j,\ i,\ 2,\ 0)\, a_j f_{j-i}' + (j,\ i,\ 2,\ 1)\, \varepsilon_j \varepsilon_{j-i}' \\
&\qquad + 2\,(i,\ 2)(a_{i-j-1} f_j + \varepsilon_{i-j-1}\varepsilon_j) + 2\,[i,\ 2](a_{-i-j-1} f_j' + \varepsilon_{-i-j-1}'\varepsilon_j')\} = 0, \\
&\Sigma_j \{ (j-i,\ -i,\ -2,\ 0)\, a_{j-i} f_j + (j-i,\ -i,\ -2,\ -2)\, a_j f_{j-i}' \\
&\qquad\qquad + (j-i,\ -i,\ -2,\ -1)\, \varepsilon_j \varepsilon_{j-i}' \\
&\quad + 2\,[-i,\ -2](a_{i-j-1} f_j + \varepsilon_{i-j-1}\varepsilon_j) + 2\,(-i,\ -2)(a_{-i-j-1} f_j' + \varepsilon_{-i-j-1}'\varepsilon_j')\} = 0.
\end{aligned}$$

The first line of the second equation has been transformed by putting $j-i$ for j in order to make it uniform for calculation with the first equation. The two differ now only in their coefficients and can be dealt with together. The coefficients in the first line of each equation are transformed in the same way as in IV. It is evident that the known value of c is sufficient, all the terms being of the order Y_0^2. The terms depending on ε_i, ε_i' are fully calculated at the first approximation, and we proceed by methods like those used in obtaining the numerical values of δa_i.

The results are as follows:

$$\frac{f_0}{Y_0^2} = +.09402\,355 \qquad \frac{f_0'}{Y_0^2} = +.03180\,170$$

$$\frac{f_{-1}}{Y_0^2} = -.06517\,276 \qquad \frac{f_1'}{Y_0^2} = +.01564\,642$$

$$\frac{f_{-2}}{Y_0^2} = +.00132\,915 \qquad \frac{f_2'}{Y_0^2} = +.00428\,597$$

$$\frac{f_{-3}}{Y_0^2} = +.00000\,174 \qquad \frac{f_3'}{Y_0^2} = +.00004\,843$$

$$\frac{f_{-4}}{Y_0^2} = +.00000\,003 \qquad \frac{f_4'}{Y_0^2} = +.00000\,049$$

$$\frac{f_1}{Y_0^2} = +.00112\,370 \qquad \frac{f_{-1}'}{Y_0^2} = +.00006\,457$$

$$\frac{f_2}{Y_0^2} = +.00001\,161 \qquad \frac{f_{-2}'}{Y_0^2} = +.00000\,066$$

$$\frac{f_3}{Y_0^2} = +.00000\,011 \qquad \frac{f_{-3}'}{Y_0^2} = +.00000\,001$$

The maximum number of approximations necessary to obtain these results was four. The common divisor of f_{-1} and f'_1 is

$$(2-2c)^2\{2(2-2c)^2-2-4m+m^2\}.$$

As $c = 1.07\ldots.$, the factor $(2-2c)^2$ causes this divisor to be small and f_{-1}, f'_1 correspondingly large. As f_{-1} and f'_1 are comparable in size with f_0 and f'_0, the number of approximations necessary for a given accuracy is much increased. The extra labor involved is not, however, very great, as very few of the terms in the expressions are affected.

When this divisor is expanded in powers of m, a series whose early terms (in the case of our moon) gradually increase, results. For

$$c-1 = m - \frac{3}{4}m^2 - \frac{201}{32}m^3 - \ldots.$$

$$\frac{1}{(c-1)^2} = \frac{1}{m^2}\left(1+\frac{3}{2}m+\frac{57}{4}m^2+\ldots.\right)$$

accounting in some measure for the slow convergence of Delaunay's coefficient of the term with argument $2D - 2l$ (see Section IX).

VII.

It is now necessary to determine the parts of $A_{i,\pm 1}$ which depend on the cube of the eccentricity: there are none dependent on the square of that quantity, as has been pointed out. The chief difficulty here arises from the fact that it is necessary to obtain a further approximation to the value of c. This latter is, however, the principal object in view in this section, and it was necessary to push the computations thus far in order to achieve it.

In equation (4) put $p = 1$ and let q take in addition such additional values that the terms depending on Y_0^3 may be included. Reference is made to Sections III and IV. Using the previous notation as far as it is available, the equation for $p = 1$ is

$$\begin{aligned}P_{j,\,i}A_{j-i,\,0}A_{j,\,1} + Q_{j+i,\,i}A_{j+i,\,0}A_{j,\,-1} &+ (j,\,i,\,1,\,2)A_{j,\,2}A_{j-i,\,1}\\ &+ (j,\,i,\,1,\,-1)A_{j,\,-1}A_{j-i,\,-2} + (i,\,1)(A_{i-j-1,\,0}A_{j,\,1} + A_{j,\,2}A_{i-j-1,\,-1})\\ &+ [i,\,1]\,(A_{-i-j-1,\,0}A_{j,\,-1} + A_{j,\,2}A_{-i-j-1,\,-1}) = 0.\end{aligned}$$

We now put as before

$$A_{i,\,0} = a_i + \delta a_i, \qquad A_{i,\,2} = f_i, \qquad A_{i.\,-2} = f'_i,$$

and now

$$A_{i,\,1}=\varepsilon_i+\delta\varepsilon_i, \qquad A_{i,\,-1}=\varepsilon'_i+\delta\varepsilon'_i,$$

so that ε_i, ε'_i denote the values found in the first part of this paper of the order Y_0 and $\delta\varepsilon_i$, $\delta\varepsilon'_i$ denote the new parts to be found of the order Y_0^3. Also let δc denote the new part of the ratio of the two periods of the order Y_0^2, so that $c+\delta c$ represents its full value to this order. Let us for a moment denote by F_i, the divisor

$$(2i+pc+p\delta c)\{2(2i+pc+p\delta c)^2-2-4m+m^2\}$$

which is common to all the coefficients. The coefficients (i, p), $[i, p]$ have in addition the divisor $(2i+pc+p\delta c)$. Multiply the equation by F_i; it will then be free from fractions involving $c+\delta c$ in the denominators except as regards the coefficients just mentioned. The equation has been already solved when δc, $\delta\varepsilon_i$, $\delta\varepsilon'_i$, δa_i and terms of higher order are put equal to zero. Hence taking the variations of c, ε_i, ε'_i, a_i and including such terms of higher orders as are required, we have, after division by F_i (in which expression δc can be put zero), for $p=1$

$$\begin{aligned}
&\{P_{j,\,i}a_{j-i}+2\,(i,\,1)\,a_{i-j-1}\}\delta\varepsilon_j+\{Q_{j+i,\,i}a_{j+i}+2\,[i,\,1]\,a_{-i-j-1}\}\delta\varepsilon'_j\\
&+\{P_{j,\,i}\delta a_{j-i}+2\,(i,\,1)\,\delta a_{i-j-1}\}\varepsilon_j+\{Q_{j+i,\,i}\delta a_{j+i}+2\,[i,\,1]\,\delta a_{-i-j-1}\}\varepsilon'_j\\
&+\left[a_{j-i}\frac{\partial}{\partial c}(P_{j,\,i}F_i)+2a_{i-j-1}\frac{\partial}{\partial c}\{(i,\,1)\,F_i\}\right]\frac{\varepsilon_j\delta c}{F_i}\\
&\qquad\qquad+\left[a_{j+i}\frac{\partial}{\partial c}(Q_{j+i,\,i}F_i)+2a_{-i-j-1}\frac{\partial}{\partial c}\{[i,\,1]\,F_i\}\right]\frac{\varepsilon'_j\delta c}{F_i}.\\
&+(j,\,i,\,1,\,2)f_j\varepsilon_{j-i}+(j,\,i,\,1,\,-1)\,\varepsilon'_jf'_{j-i}+2\,(i,\,1)f_j\varepsilon'_{i-j-1}+2\,[i,\,1]\varepsilon_jf'_{-i-j-1}=0.
\end{aligned}$$

Here Σ_j is to be understood to stand before each term. There will be a similar equation for $p=-1$, $\delta\varepsilon_i$ and $\delta\varepsilon'_{-i}$ being a pair which are calculated together.

The method of proceeding is by continued approximation as in the determination of ε_i, ε'_i. We obtain $\delta\varepsilon_i$, $\delta\varepsilon'_i$ for a sufficient number of values of i, *excluding* $i=0$, in terms of $\delta\varepsilon_0$, $\delta\varepsilon'_0$, $Y_0\delta c$, Y_0^3, the results as is evident being linear functions of these quantities with numerical coefficients. The results are then substituted in the two equations given by $i=0$ (or two equivalent equations) and we obtain two linear relations between the same four quantities.

The number of terms to be computed is very large, but the following observations will show that the amount of calculation is really smaller than at first sight appears.

There are four sets of terms involving in combination (i) the letters a, $\delta\varepsilon$; (ii) the letters ε, δa; (iii) a, ε, δc; (iv) ε, f. These can be dealt with separately at the first approximation.

In set (i) we are going to determine the parts of $\delta\varepsilon_i$, $\delta\varepsilon_i'$ which depend on $\delta\varepsilon_0$, $\delta\varepsilon_0'$. But the expressions are exactly the same as those given in IV for determining ε_i, ε_i' in terms of ε_0, ε_0'. The results are therefore to hand.

In set (ii) the coefficients are the same as those in IV and are therefore known; also as δa_i, ε_i, ε_i' are known, the set can be completely calculated at the first approximation in terms of Y_0^3.

In set (iii) the new coefficients have to be found, a_i, ε_i, ε_i' are known. These terms can also be completely obtained at the first approximation, expressed in terms of $Y_0\delta c$; as δc comes out approximately .0027 Y_0^3, these terms are calculated to two fewer places of decimals than those involving Y_0^3.

Set (iv) can be calculated completely at the first approximation in terms of Y_0^3, as all the quantities are known. In addition the coefficients $(i, 1)$, $[i, 1]$ have been found, while as

$$(j, i, 1, q) = (2j + qc) L_i + (2j + qc)^2 M_i,$$

and that L_i, M_i have been calculated, these coefficients are soon obtained.

Having calculated as far as possible all the terms at the first approximation, we use the values of $\delta\varepsilon_i$, $\delta\varepsilon_i'$ thus obtained for the second approximation. Set (i) will be the only one affected, and its coefficients have been found. These approximations will of course add to the coefficients of Y_0^3 and $Y_0\delta c$, but not to those of $\delta\varepsilon_0$ and $\delta\varepsilon_0'$, as is evident from the remarks on set (i).

The coefficients ε_i, ε_i' have been expressed in terms of Y_0 or $\varepsilon_0 - \varepsilon_0'$. It now appears that Y_0 has an additional part $\delta\varepsilon_0 - \delta\varepsilon_0'$ to be added to it. This really amounts to introducing a new arbitrary constant, or, in other words, to making $\varepsilon_0 - \varepsilon_0' + \delta\varepsilon_0 - \delta\varepsilon_0'$ the arbitrary constant instead of $\varepsilon_0 - \varepsilon_0'$. This is inconvenient and causes unnecessary complications. Since we are at liberty to choose our arbitrary, it will be simplest to make it Y_0, so that $\delta Y_0 = 0$. Hence the coefficient of the principal elliptic term in the expression for the coordinate y will, in this theory, be

$$a_0 + \delta a_0 Y_0,$$

where $a_0 + \delta a_0$ has the value which is found above expressed in terms of $(\mu/n^2)^{\frac{1}{3}}$ and Y_0^2. We have then $\delta\varepsilon_0' = \delta\varepsilon_0$.

The values of $\delta\varepsilon_i$, $\delta\varepsilon_i'$ found as above are given below.

$$\begin{aligned}
\delta\varepsilon_{-1} &= +.02435\ 517\ Y_0^3 &&+.22567\ 789\ \delta\varepsilon_0 &&+.76306\ 1\ Y_0\delta c,\\
\delta\varepsilon_1' &= -.00877\ 697\ Y_0^3 &&-.08834\ 235\ \delta\varepsilon_0 &&-.24185\ 2\ Y_0\delta c,\\
\delta\varepsilon_1 &= +.02529\ 357\ Y_0^3 &&+.00215\ 229\ \delta\varepsilon_0 &&-.00006\ 1\ Y_0\delta c,\\
\delta\varepsilon_{-1}' &= +.00322\ 634\ Y_0^3 &&-.00128\ 260\ \delta\varepsilon_0 &&+.00079\ 8\ Y_0\delta c,\\
\delta\varepsilon_{-2} &= +.00070\ 947\ Y_0^3 &&+.00008\ 499\ \delta\varepsilon_0 &&+.00003\ 7\ Y_0\delta c,\\
\delta\varepsilon_2' &= +.00526\ 427\ Y_0^3 &&-.00049\ 147\ \delta\varepsilon_0 &&-.00145\ 3\ Y_0\delta c,\\
\delta\varepsilon_2 &= +.00048\ 131\ Y_0^3 &&+.00000\ 620\ \delta\varepsilon_0 &&-.00000\ 1\ Y_0\delta c,\\
\delta\varepsilon_{-2}' &= +.00002\ 534\ Y_0^3 &&-.00000\ 076\ \delta\varepsilon_0,\\
\delta\varepsilon_{-3} &= +.00000\ 493\ Y_0^3 &&-.00000\ 019\ \delta\varepsilon_0,\\
\delta\varepsilon_3' &= +.00009\ 895\ Y_0^3 &&-.00000\ 333\ \delta\varepsilon_0 &&-.00001\ 0\ Y_0\delta c,\\
\delta\varepsilon_3 &= +.00000\ 678\ Y_0^3 &&-.00000\ 001\ \delta\varepsilon_0,\\
\delta\varepsilon_{-3}' &= +.00000\ 031\ Y_0^3,\\
\delta\varepsilon_{-4} &= +.00000\ 006\ Y_0^3,\\
\delta\varepsilon_4' &= +.00000\ 137\ Y_0^3.
\end{aligned}$$

In order to obtain the numerical ratios of $\delta\varepsilon_0$, δc to Y_0^3, we may use the two equations of the system just considered for $i = 0$. As will be stated below, an equation of verification was, however, computed,—in this case, the second of equations (6) in the first part of this paper, with the necessary terms of order Y_0^3 included. These three equations gave respectively

$$\begin{aligned}
+.00425\ 226\ Y_0^3 + .02946\ 89\ \delta\varepsilon_0 - 1.11898\ 7\ Y_0\delta c &= 0\\
+.00902\ 093\ Y_0^3 + .00129\ 30\ \delta\varepsilon_0 - 3.33860\ 6\ Y_0\delta c &= 0\\
+.18016\ 812\ Y_0^3 + 4.32005\ 66\ \delta\varepsilon_0 + .98781\ 5\ Y_0\delta c &= 0.
\end{aligned}$$

The first of these combined with the third gave

$$\delta c = +.00268\ 561\ Y_0^2,$$

and the second with the third gave 2 instead of 1 for the last place of decimals. Whence

$$\delta\varepsilon_0 = \delta\varepsilon_0' = -.04231\ 912\ Y_0^3.$$

Substituting in the table just given for the values of $\delta\varepsilon_i$, we obtain as the final values for these

$$\begin{array}{llll}
\dfrac{\delta\varepsilon_{-1}}{Y_0^3}=+.01685\ 40 & \dfrac{\delta\varepsilon'_1}{Y_0^3}=-.00568\ 79 & \dfrac{\delta\varepsilon_1}{Y_0^3}=+.02520\ 23 & \dfrac{\delta\varepsilon'_{-1}}{Y_0^3}=+.00328\ 28 \\
\dfrac{\delta\varepsilon_{-2}}{Y_0^3}=+.00070\ 60 & \dfrac{\delta\varepsilon'_2}{Y_0^3}=+.00528\ 12 & \dfrac{\delta\varepsilon_2}{Y_0^3}=+.00048\ 10 & \dfrac{\delta\varepsilon'_{-2}}{Y_0^3}=+.00002\ 54 \\
\dfrac{\delta\varepsilon_{-3}}{Y_0^3}=+.00000\ 49 & \dfrac{\delta\varepsilon'_3}{Y_0^3}=+.00009\ 91 & \dfrac{\delta\varepsilon_3}{Y_0^3}=+.00000\ 68 & \dfrac{\delta\varepsilon'_{-3}}{Y_0^3}=+.00000\ 03 \\
 & \dfrac{\delta\varepsilon'_4}{Y_0^3}=+.00000\ 14 & &
\end{array}$$

It is not my intention to carry the calculation of the coefficients to any further extent. Enough has been given to show how they can be obtained to any desired accuracy; and the methods outlined above for making the computations can be arranged so that the assistance of a computer of ordinary intelligence can be largely made use of. One of the chief advantages of these methods is the ease with which equations of verification can be employed. Before comparing some of the results with the corresponding ones of Delaunay, I shall give a short account of the methods of verification used.

VIII.

Verification of the Results.

There appear to be two principal sources of error in the class of computations here treated apart from that produced by the use of logarithms. The first is the common one of making an actual numerical mistake of any kind whatever; the second arises from a term omitted altogether. It is therefore necessary to have such equations of verification as shall give some security that errors proceeding from *both* these sources may be eliminated. Owing to the interdependence of the various approximations and, in the case of the higher coefficients, to the large number of terms to be computed, it is advisable to have at each step a test for errors arising from the first and chief source. For this object the equation most useful, on account of its simplicity, is the second of equations (3). It was used in all the calculations for coefficients depending on the square and cube of the eccentricity. In it the terms are arranged in the same way as those in the equations used for finding the results; that is to say, the suffixes proceed in the same order, and the coefficients depending on m, c alone differ. An error therefore of a sensible term omitted altogether in finding any pair of coefficients would probably also occur in the equation of verification. Such a case occurred once in these computations. For the coefficients depending on the first power of the eccentricity, one of equations (6) was used, as the computations of the coefficients of its various terms were simple, and as it served at one calculation to verify the four quantities of the same order, ε_i, ε'_{-i}, ε'_i, ε_{-i}, while the second of equations (3) required two calculations, one for ε_i, ε'_{-i} and another for ε'_i, ε_{-i}. It will be noticed, too, that the terms in equations (6) are arranged somewhat differently, a property which possesses some value in this connection.

To detect an error of the second kind, reliance was placed on the computation of an expression which has an entirely different form. In the equation

$$uD^2s + sD^2u + Du.Ds - 2m(uDs - sDu) + \frac{9m^2}{4}(u+s)^2 = C$$

substitute the values of u, s, Du, etc., found and arrange the result in powers of Y_0 and $\zeta^{\pm c}$. The result will be of the form

$$\frac{C}{(a_0+\delta a_0)^2} = \alpha + \alpha' Y_0^2 + (\beta Y_0 + \beta' Y_0^3)(\zeta^c + \zeta^{-c}) + \gamma Y_0^2(\zeta^{2c} + \zeta^{-2c}) + \dots .$$

Here α, α', β, β', γ will be series in ζ^{2i} with known numerical coefficients. As this equation should be true for *all* values of ζ, we should have

$$\frac{C}{(a_0+\delta a_0)^2} = \alpha + \alpha' Y_0^2, \quad \beta Y_0 + \beta' Y_0^3 = 0, \quad \gamma = 0$$

to the degree of approximation used; moreover, since Y_0 is arbitrary, in addition we have

$$\beta = 0, \quad \beta' = 0.$$

Again, as these equations ought to hold for all values of ζ, we can put $\zeta = 1$ in each of them; the verification depends on the results being identically zero. Its value arises from the fact that the expressions to be here calculated are all of such forms as

$$\Sigma_i a_i, \quad \Sigma(2i+1)a_i, \quad \Sigma(2i+1+c)^2\varepsilon_i, \quad \text{etc.},$$

they are therefore totally different from any of those previously used. The higher coefficients as $\delta\varepsilon_3$, f_4, etc., are, however, multiplied by large numbers in some cases, and therefore unless these are carried to a higher degree of approximation than was before found necessary, this method fails to test quite fully the results obtained. For practical purposes, since the coefficients have been computed to a large number of places of decimals, the test appears to be sufficient.

The Jacobian constant is obtained by putting $p=0$ in the first of equations (3). Let C now denote the initial value of this constant as found by Dr. Hill, and δC the new part depending on Y_0^2. From the equation we obtain by means of the known values of δa_i,

$$\frac{\delta C}{a_0^2} = +.34552\ 75\ Y_0^2.$$

The value of δC obtained from the equation of verification above where ζ is put equal to unity, differs only by one unit in the last place. This therefore gives the required verification for the series of coefficients δa_i. The coefficients f_i, f_i' are absent from these expressions.

The case of these coefficients δa_i requires special mention. The two values of δC found from two entirely independent expressions differ only by one unit in the seventh figure. That is to say, they are the same within the limits of error. It seems very improbable that an error of one unit in the fourth significant figure of δa_1 or δa_{-1}, or of the first significant figure in δa_2 or δa_{-2}, should have been able to elude detection while running through two separate sets of equations of verification. Yet this is roughly the amount of error required to make the results agree with those of Delaunay as shown in the next section.

In order to verify the transformations from rectangular to polar coordinates, in which the method of 'special values' was adopted, an extra 'special value' was computed and the results were found to be accordant.

All computations once made were gone through a second time. The average error made in the later portions of the work was about one in every four or five hundred figures. It did not seem to be confined to any particular class of operation. In using the new eight-figure tables of the French Government, extra care was exercised for the differences, and the chance of error thus diminished.

IX.

Transformation to Polar Coordinates.

Let v denote the difference between the true and mean longitudes. Gathering together the results of the previous sections and adding those given by Dr. Hill (Vol. I, p. 248), we have with Delaunay's notation for the arguments; that is, $2D$ for that of the 'Variation' and l for that of the principal elliptic term:

$$\begin{aligned}
\frac{r\cos v}{a_0+\delta a_0} = 1 &+ (-.00718\;00395 + .05314\;689\;\; Y_0^2)\cos 2D \\
&+ (+.00000\;60424 + .00048\;328\;\; Y_0^2)\cos 4D \\
&+ (+.00000\;00325 + .00000\;499\;\; Y_0^2)\cos 6D \\
&+ (+.00000\;00002 + .00000\;005\;\; Y_0^2)\cos 8D \\[1ex]
&+ (-.49679\;1802\;\; Y_0 - .08463\;82\;\; Y_0^3)\cos l \\
&+ (-.09332\;9284\;\; Y_0 + .01116\;60\;\; Y_0^3)\cos(2D-l) \\
&+ (+.00025\;6338\;\; Y_0 + .00598\;71\;\; Y_0^3)\cos(4D-l) \\
&+ (+.00000\;2210\;\; Y_0 + .00010\;40\;\; Y_0^3)\cos(6D-l) \\
&+ (+.00000\;0025\;\; Y_0 + .00000\;14\;\; Y_0^3)\cos(8D-l) \\
&+ (+.00134\;2824\;\; Y_0 + .02848\;51\;\; Y_0^3)\cos(2D+l) \\
&+ (+.00001\;0769\;\; Y_0 + .00050\;64\;\; Y_0^3)\cos(4D+l) \\
&+ (+.00000\;0079\;\; Y_0 + .00000\;71\;\; Y_0^3)\cos(6D+l) \\[1ex]
&+ .12582\;524\;\; Y_0^2\cos 2l \\
&- .04952\;634\;\; Y_0^2\cos(2D-2l) + .00118\;827\;\; Y_0^2\cos(2D+2l) \\
&+ .00561\;512\;\; Y_0^2\cos(4D-2l) + .00001\;227\;\; Y_0^2\cos(4D+2l) \\
&+ .00005\;017\;\; Y_0^2\cos(6D-2l) + .00000\;012\;\; Y_0^2\cos(6D+2l) \\
&+ .00000\;051\;\; Y_0^2\cos(8D-2l).
\end{aligned}$$

$$\begin{aligned}
\frac{r\sin v}{a_0+\delta a_0} = &+ (+.01021\;14544 + .02561\;651\;\; Y_0^2)\sin 2D \\
&+ (+.00000\;57149 + .00043\;897\;\; Y_0^2)\sin 4D \\
&+ (+.00000\;00276 + .00000\;447\;\; Y_0^2)\sin 6D \\
&+ (+.00000\;00002 + .00000\;005\;\; Y_0^2)\sin 8D
\end{aligned}$$

$$
\begin{aligned}
&+(+1.00000\;0000\;Y_0 \qquad\qquad\qquad\quad)\sin l\\
&+(+\;\;.20446\;5776\;Y_0-.02254\;19\;\;Y_0^3)\sin(2D-l)\\
&+(+\;\;.00036\;0509\;Y_0+.00457\;52\;\;Y_0^3)\sin(4D-l)\\
&+(+\;\;.00000\;1960\;Y_0+.00009\;41\;\;Y_0^3)\sin(6D-l)\\
&+(+\;\;.00000\;0023\;Y_0+.00000\;13\;\;Y_0^3)\sin(8D-l)\\
&+(+\;\;.00159\;6237\;Y_0+.02191\;96\;\;Y_0^3)\sin(2D+l)\\
&+(+\;\;.00000\;9426\;Y_0+.00045\;57\;\;Y_0^3)\sin(4D+l)\\
&+(+\;\;.00000\;0069\;Y_0+.00000\;65\;\;Y_0^3)\sin(6D+l)
\end{aligned}
$$

$$
\begin{aligned}
&+.06222\;185\;\;Y_0^2\sin 2l\\
&+.08081\;918\;\;Y_0^2\sin(2D-2l)+.00105\;914\;\;Y_0^2\sin(2D+2l)\\
&+.00295\;682\;\;Y_0^2\sin(4D-2l)+.00001\;094\;\;Y_0^2\sin(4D+2l)\\
&+.00004\;669\;\;Y_0^2\sin(6D-2l)+.00000\;011\;\;Y_0^2\sin(6D+2l)\\
&+.00000\;046\;\;Y_0^2\sin(8D-2l).
\end{aligned}
$$

To find the longitude we can proceed as follows. The difference of the true and mean longitudes is denoted by

$$v=\tan v-\tfrac{1}{3}\tan^3 v+\tfrac{1}{5}\tan^5 v-\ldots.$$

Tan v is obtained from the above values and the terms are expanded in powers of Y_0. To Y_0^0 are attached the sines of the arguments $2iD$, to Y_0^1 the sines of $2iD\pm l$, and to Y_0^2 those of $2iD$, $2iD\pm 2l$. The coefficients depending on Y_0^1 have already been obtained and discussed in the first part of this paper. As far as the order Y_0^2 let

$$
\begin{aligned}
r\cos v&=S_0+S_1Y_0+S_2Y_0^2\\
r\sin v&=S_0'+S_1'Y_0+S_2'Y_0^2
\end{aligned}
$$

where S_0, S_1, etc., denote the terms involving the respective powers of Y_0, etc., in the expressions just given. It is not difficult to prove that when these are substituted in the expansion for v, we obtain for the terms involving Y_0^2 in v

$$\frac{S_0S_2'-S_0'S_2}{S_0^2+S_0'^2}+\frac{S_0S_0'(S_1^2-S_1'^2)-S_1S_1'(S_0^2-S_0'^2)}{(S_0^2+S_0'^2)^2}.$$

The arguments $\pm\,2l$ only appear in the numerators. The coefficients in this expression of $\cos 2l$, $\sin 2l$, and that in which $2l$ does not appear, are quickly separated out and the results are series involving sines and cosines of the arguments $2iD$. To each of these three sets of terms the method of special values for $2D$ can then be applied and the required coefficients of the terms in longitude obtained.

The results for these terms are:

$$\begin{aligned}+ Y_0^2 [&+ .12015\,93 \sin 2D + .30993\,73 \sin 2l + .08545\,32 \sin (2D - 2l)\\ &+ .00209\,70 \sin 4D + .00532\,16 \sin (2D + 2l) + .01248\,18 \sin (4D - 2l)\\ &+ .00002\,70 \sin 6D + .00006\,81 \sin (4D + 2l) + .00021\,37 \sin (6D - 2l)\\ &+ .00000\,03 \sin 8D + .00000\,07 \sin (6D + 2l) + .00000\,27 \sin (8D - 2l)].\end{aligned}$$

The relation between Y_0 and Delaunay's e has been found in Section IV By means of it these coefficients expressed in seconds of arc become

$$\begin{aligned}&+ 298''.959 \sin 2D + 771''.132 \sin 2l + 212''.610 \sin (2D - 2l)\\ &+ 5''.217 \sin 4D + 13''.240 \sin (2D + 2l) + 31''.055 \sin (4D - 2l)\\ &+ 0''.067 \sin 6D + 0''.169 \sin (4D + 2l) + 0''.532 \sin (6D - 2l)\\ &+ 0''.001 \sin 8D + 0''.002 \sin (6D + 2l) + 0''.007 \sin (8D - 2l).\end{aligned}$$

The part of the motion of the Lunar Perigee which depends on Y_0^2 has been found above to be

$$\delta c = + .00268\,561\, Y_0^2;$$

and the relation between Y_0 and Delaunay's eccentricity,

$$Y_0 = 2e \times 1.00027\,136.$$

Also
$$\delta \left(\frac{1}{n} \cdot \frac{d\omega}{dt}\right) = - \frac{1}{1 + m}\, \delta c;$$

hence
$$\delta \left(\frac{1}{n} \cdot \frac{d\omega}{dt}\right) = - .00994\,29\, e^2.$$

The series given by Delaunay for this quantity is*

$$- e^2 \left[\frac{3}{8} m_1^2 + \frac{675}{64} m_1^3 + \frac{31605}{512} m_1^4 + \frac{1483665}{4096} m_1^5 + \frac{25291729}{16384} m_1^6 - \frac{352038885}{1179648} m_1^7\right],$$

---

*Comptes Rendus, Tom. LXXIV, p. 19.

where $m_1 = n'/n$. In numbers the portion within the brackets becomes

$$.00209\ 82 + .00441\ 42 + .00193\ 25 + .00084\ 82 + .00027\ 04 - .00000\ 39 = .00955\ 96,$$

which is considerably less than the previous value, the difference being greater than the last term but one of this series. Moreover, Delaunay's series seems to indicate that when estimation is made for the uncalculated portion, his value should be even less still. The results, therefore, can scarcely be considered accordant. The difference amounts to $0''.055$ in the motion.

The coefficients found above agree with what might have been expected from Delaunay's series, with one exception, the part of the coefficient of the variation which depends on e^2. Expressed in seconds of arc, Delaunay's values for the terms e^2m, e^2m^2, etc., in the coefficient are*

$$217''.976 + 59''.839 + 16''.331 + 3''.852 + 0''.777 + 0''.137 - 0''.049 = 298''.863;$$

and it would appear from this that the true value was somewhat less, say

	$298''.84$
the value obtained above is	$298''.96$
a difference of	$0''.12$

It should be pointed out that in Prof. Newcomb's comparison of Delaunay's and Hansen's Lunar Theories,† there seems no reason to expect that the former's value of the coefficient is erroneous by so much as one-tenth of a second. It may, however, have happened that there is an error in some other part of the long expression for this coefficient which may balance a possible error here, or, what is more likely, that the other terms if carried further would have given some considerable coefficient. The terms in e'^2 converge very slowly up to the last given.

It may be mentioned here, that in comparing the results for many of the coefficients with those given by Delaunay the differences are less than a hasty inspection would have caused us to estimate. If we examine the series arranged in powers of m, more carefully, and especially those which have a power of the eccentricity as a factor, it appears that in most cases the terms have a tendency sooner or later to change sign and the term before the change of sign has a comparatively small numerical multiplier. Delaunay sometimes stops at this term, thus giving a false idea of the accuracy of the total result.

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* Mém. de l'Acad. des Sc., Tom. XXIX, p. 815.

† Astron. Papers of Amer. Eph., Vol. I, p. 92.

The most serious difference is that in the value of the part of the Lunar Perigee just obtained, depending on e^2 and the ratio of the mean motions only, since, as Dr. Hill has pointed out,* the motion of the Perigee is capable of being found observationally with a high degree of accuracy. It is seen from the expansion reproduced above that Delaunay in this very slowly converging series has carried his terms beyond the change of sign, and another change of sign in the uncalculated portion, which would be necessary to make the two results accordant, seems hardly likely to occur early enough to produce the difference. In either theory the difference, just mentioned, in the coefficient of sin $2D$ would be sufficient to cause the discordance. As I have said in Section VIII, it seems hardly probable that a sufficient error has been made in the calculations undertaken to obtain δa_i. It might be surmised that the error arises from the set $\delta \varepsilon_i$, $\delta \varepsilon_i'$ The error required there would be several units in the third place of decimals, and again this seems improbable in view of the two sets of equations of verification.

One other coefficient is worthy of special notice, namely that of sin $(2D - 2l)$. Delaunay gets for this†

$$130''.786 + 46''.089 + 22''.283 + 8''.131 + 3''.179 + 1''.308 + 0''.542$$
$$= 212''.318$$

my coefficient being $212''.610$

so that the remainder of Delaunay's series is $+0''.29$. The values of f_i, f_i' not depending on those of δa_i and vice versâ, an error in one set does not involve one in the other.

In a similar way may the parts depending on Y_0^3 be transformed to polar coordinates. The coefficient of sin l in longitude will be of the form

$$\alpha Y_0 + \beta Y_0^3$$

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[*] On the Part of the Motion of the Lunar Perigee, etc. Acta Math., Vol. VIII, p. 1.

[†] Mem. cit., p. 823.

where α, β are numerical coefficients. In Delaunay's theory this coefficient is

$$2e - \tfrac{1}{4}e^3,$$

and therefore the second approximation to Y_0 in terms of e will be given by

$$\alpha Y_0 = 2e - \tfrac{1}{4}e^3 - \frac{8\beta}{\alpha^3} e^3$$

This must be applied to all terms with arguments $2iD \pm l$ when it is desired to express them in Delaunay's notation.

HAVERFORD COLLEGE, *June*, 1893.